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Existence of Admissible Rational Chebyshev Approximations on Subsets

Communicated by E. W. Cheney

The problem of rational approximation on compact sets is discussed in [1]. A rational function $R(A) = P(A, \cdot)/Q(A, \cdot)$ is called *admissible* on X if $Q(A, x) > 0$ for all $x \in X$. For a given f there may not exist a best approximation on X by admissible rational functions, and existence on X does not guarantee existence on finite subsets. If $\{A_k\} \to A$ as $k \to \infty$ and if $R(A)$ is admissible, then $R(A_k)$ is also admissible for all sufficiently large k. Thus we can append to Corollaries 2 and 3 of [1] the words "and $R(A_k)$ is admissible for all sufficiently large k."

Recall now that the density of a set Y in a metric space X is defined to be $\max_{y \in Y} \min_{x \in X} d(x, y)$.

THEOREM. *Let X be a metric space, and let f be a function which has a unique best admissible approximation $R(A)$. Assume that $R(A)$ cannot be represented as $R(C)$, where $Q(C, \cdot)$ has a zero. Then there exists a $\delta > 0$ such that for any compact Y of density less than δ, each best approximation to f on Y is admissible on X.*

Proof. If the result is false, then there exist sets $X_1, X_2, \ldots$ such that: (1) for any $x \in X$ there are points $x_k \in X_k$ converging to x, and (2) there is a best approximation $R(A_k)$ to f on X_k which is not admissible. This contradicts the modified Corollary 2.

A special case of this theorem can be obtained by replacing the first sentence by "Let f have $R(A)$ as an admissible best approximation, and assume that $S(A)$ is an $(n + m - 1)$-dimensional Haar subspace." In the case of polynomial rational functions on an interval $[\alpha, \beta]$, we have the following corollary.

COROLLARY. *Let f have a nondegenerate best approximation by polynomial rational functions from $R_m^n[\alpha, \beta]$. Then there exists $\delta > 0$ such that if the density of Y is less than δ, there is a (unique) best approximation by admissible rationals to f on Y.*

Proof. Let $R(A)$ be best from $R_m^n[\alpha, \beta]$ on $[\alpha, \beta]$. As it is nondegenerate, $S(A)$ is a Haar subspace of dimension $n + m - 1$. By our previous results, any best approximation by rationals on Y is admissible on $[\alpha, \beta]$. It is unique as a best approximation from $R_m^n[\alpha, \beta]$ on Y is unique.

The following example shows that the representation or nondegeneracy hypotheses of the previous results cannot be deleted.

EXAMPLE. Let $X = [-1, 1]$, $f(x) = x$, $R(A, x) = a_1/(a_2 + a_3 x)$, $X_k = \{-1 + 1/k, -1 + 2/k, \ldots, (k-1)/k, 1\}$. As f alternates once on X, 0 is the unique best approximation to f on X. There is no best admissible approximation to f on X_k. If it existed, it would have to be 1 at 1 and zero on the rest of X_k. But such a function is not admissible.

REFERENCE

1. C. B. DUNHAM, Rational Chebyshev Approximation on Subsets, *J. Approximation Theory* **1** (1968), 484–487.

CHARLES B. DUNHAM

Computer Science Department
University of Western Ontario
London, Canada

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